

Second order LDE with variable coefficients.

$$\frac{d^2 y}{dx^2} + p(x) \frac{dy}{dx} + q(x)y = r(x) \quad (1)$$

If in (1), $r(x) = 0$ Homog. eqⁿ.

$r(x) \neq 0$ Non-homog. eqⁿ.

It will have two sol., say y_1 & y_2 are sol. of homog. second order LDE, then $y = c_1 y_1 + c_2 y_2$ is also a sol. of same D.E. ($c_1, c_2 \in \mathbb{R}$)

The set $\{y_1, y_2\}$ is fundamental set of sol. if y_1, y_2 are L.I. on open interval I .

L.I.: $f_1(x)$ & $f_2(x)$ are L.I. on open interval I if
 $c_1 f_1(x) + c_2 f_2(x) \neq 0 \Rightarrow c_1 \neq c_2 = 0$

Eg 1) $f_1(x) = \sin 2x$ $f_2(x) = \sin x \cos x$ on $(-\infty, \infty)$
 $1 \cdot \sin 2x - 2 \cdot \sin x \cos x \neq 0$ $c_1 = 1$
 $c_2 = -2 \neq 0$

2) $f_1(x) = x$ $f_2(x) = |x|$ on $(0, \infty)$
 $f_1(x) - f_2(x) = x - |x|$ on $(0, \infty)$
 $= x - x$ on $(0, \infty)$
 $= 0$
 for $c_1 = 1$
 $c_2 = -1$
 $\therefore x$ & $|x|$ are L.D. on $(0, \infty)$

$f_1(x) - f_2(x) = x - |x|$ on $(-\infty, \infty)$
 $= x - \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$
 $= \begin{cases} 0 & x \geq 0 \\ 2x & x < 0 \end{cases}$

x & $|x|$ are L.I. on $(-\infty, \infty)$

Wronskian : of 2 diff fⁿ. $y_1(x)$ & $y_2(x)$ is

$$W(y_1, y_2) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$$

$$W(y_1, y_2, y_3, y_4) = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1''' & y_2''' & y_3''' & y_4''' \end{vmatrix}$$

Q. show that sol. $y_1 = \sin x$ & $y_2 = \cos x$ of $\frac{d^2y}{dx^2} + y = 0$ are L.I.

$$W(y_1, y_2) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}$$

$$= -(\sin^2 x + \cos^2 x) = -1 \neq 0$$

Note: 1 Two sol. y_1 & y_2 of D.E. $y'' + p(x)y' + q(x)y = r(x)$ are L.I. iff $W(y_1, y_2) \neq 0$ for some $x_0 \in I$.

2) If $W = 0$ for some point $x_0 \in I$, then $W = 0 \forall x \in I$.

3) If $W \neq 0$ at some point $x_1 \in I$, then y_1 & y_2 are L.I. on that interval I .

method of reduction of order: To solve $y'' + p(x)y' + q(x)y = r(x)$

if we know one solution $f_1(x)$, then we can find 2nd solution using formula

$$f_2(x) = f_1(x) \int \frac{e^{-\int p(x) dx}}{f_1^2(x)} dx$$

Q. $(x^2 + 1)y'' - 2xy' + 2y = 0$ Give, $y = x$ is a sol.

$$f_1(x) = x$$

$$f_2(x) = x \int \frac{e^{-\int \frac{-2x}{x^2+1} dx}}{x^2} dx$$

$$x^2+1 = t$$

$$2x dx = dt$$

$$= x \int \frac{e^{\int \frac{1}{t} dt}}{x^2} dx$$

$$= x \int \frac{e^{\log(x^2+1)}}{x^2} dx$$

$$= x \int \frac{x^2+1}{x^2} dx$$

$$= x \int \left(1 + \frac{1}{x^2}\right) dx$$

$$= x \left[x - \frac{1}{x} \right] = x^2 - 1$$

G.S. is $y = C_1 f_1(x) + C_2 f_2(x) = C_1 x + C_2 (x^2 - 1)$ Ans.

$$W(f_1, f_2) = \begin{vmatrix} x & x^2-1 \\ 1 & 2x \end{vmatrix} = 2x^2 - (x^2-1) = x^2+1 > 0 \neq 0$$